ON LOGICAL CHARACTERIZATION OF HENSELIANITY

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Abstract. We give some sufficient conditions under which any valued field that admits quantifier elimination in the Macintyre language is henselian. Then, without extra assumptions, we prove that if a valued field of characteristic $(0, 0)$ has a $\mathbb{Z}$-group as its value group and admits quantifier elimination in the main sort of the Denef-Pas style language $\mathcal{L}_{RRP}$ then it is henselian. In fact the proof of this suggests that a quite large class of Denef-Pas style languages is natural with respect to henselianity.

§1. **Introduction.** One of the most important tools in model-theoretic algebra is quantifier elimination (QE). Tarski's Theorem laid the foundation for the subsequent work along this line:

THEOREM 1.1 (Tarski). *The theory RCF of real closed fields, as formulated in the language $\mathcal{L}_{OR}$ of ordered rings, admits QE.*

Much later Macintyre proved a very important analog of this result for p-adic fields in [9]:

THEOREM 1.2 (Macintyre). *The theory of p-adic fields, as formulated in the language $\mathcal{L}_{Mac}$, admits QE.*

A crucial question for the algebraic structure of a field is of course under what conditions polynomials have roots. Properties that answer this question in real closed fields and p-adic fields are essential to the proofs of the above two theorems. They are of course real-closedness and henselianity, respectively. One may raise the question: Is QE equivalent to these properties after all? For real closed fields there is a good answer:

THEOREM 1.3 (Macintyre, McKenna, van den Dries). *Let K be an ordered field such that the theory of K in $\mathcal{L}_{OR}$ admits QE. Then K is real closed.*

This result is established in [10], in which the authors actually give a quite general technique that can be used to establish other similar "converse QE" results for various kinds of fields. In particular they have the following analogous result for p-fields:

THEOREM 1.4 (Macintyre, McKenna, van den Dries). *Let K be a p-field such that the theory of K in $\mathcal{L}_{Mac}$ admits QE. Then K is p-adically closed.*

The definition of a p-field K is rather special: it is a substructure of a p-adically closed field L (of p-rank 1) with respect to $\mathcal{L}_{Mac}$. The point is that, as L is henselian, each nth power predicate P_n defines a clopen subset of K in the

1

valuation topology of K, which is essential to the proof of the theorem. This way to interpret each P_n is obviously unsatisfactory since an element in P_n may not be an nth power at all in K. Hence it is asked in [10] to extend the result to the class of valued fields where P_n is simply interpreted as the group of nth powers. In Section 3 we shall give some sufficient conditions under which any such valued field that admits QE in $\mathcal{L}_{Mac}$ is henselian. This addresses a question in [1]. In fact this result holds for certain finitely generated valued fields without QE; see Section 4.

There are variations and extensions of $\mathcal{L}_{Mac}$ in which QE results for larger classes of valued fields have been obtained, for example, [2, 12]. There are yet more languages which give rise to different techniques of QE in valued fields and which cannot be subsumed under the Macintyre style. The most notable among these is the Denef-Pas style, a mature form of which is given in [11]. In Section 5 we shall show that any valued field that admits QE in the main sort in the prototypical Denef-Pas language $\mathcal{L}_{RRP}$, which is introduced in [11], is henselian. In fact the proof of this suggests that the result holds for a quite large class of Denef-Pas style languages. This answers a question mentioned in [1].

Finally in Section 6 a general perspective on QE and converse QE results is described.

§2. Preliminaries. In this paper all valued fields are of characteristic 0 and all valuation rings are proper subrings. We use $\mathcal{O}$, $\mathcal{O}_1$, etc. and $\mathcal{M}$, $\mathcal{M}_1$, etc. to denote valuation rings and their maximal ideals, respectively. Valuation maps are denoted by v, v_1, etc. If v is a valuation of K then vK, $\overline{K}$ stand for the corresponding value group and residue field, respectively.

The Macintyre language $\mathcal{L}_{Mac}$ for valued fields contains the language of rings $\mathcal{L}_R$, $\{+, -, \cdot, 0, 1\}$, a unary predicate $\mathcal{O}$ for valuation rings, and unary predicates P_n for all $n > 1$, which are usually interpreted as the sets of nonzero nth powers.

DEFINITION 2.1. Let d be a fixed natural number. A *p-adically closed field of p-rank d* is a valued field such that

1. the value group is a $\mathbb{Z}$-group with least positive element 1;
2. the dimension of the $\mathbb{F}_p$-module $\mathcal{O}/(p)$ is d, which is to say that the residue field is a finite extension of $\mathbb{F}_p$ of dimension f, $v(p) = e \cdot 1$ for some $e \in \mathbb{N}$, and $d = e \cdot f$;
3. Hensel's Lemma holds.

Prestel and Roquette extended Theorem 1.2 to the class of p-adically closed fields of finite p-ranks, providing that for each p-rank d one expands $\mathcal{L}_{Mac}$ by adding d new constants that serve as a $\mathbb{F}_p$-basis of $\mathcal{O}/(p)$; see [12, Theorem 5.6].

The proof of Thereom 1.4 relies on the approximation technique devised in [10]. In general this technique consists of the following three steps. Let (K, v) be a valued field such that $\mathrm{Th}(K)$ admits QE (in the main sort) in some language for valued fields, where $\mathrm{Th}(K)$ denotes the theory of K as a structure of the language in question. Let $\mathcal{O}, \mathcal{M}$ be its valuation ring and maximal ideal. For convenience, throughout this paper, by valuation topology we mean the topology on $K^\times$ (instead of K) that is induced by the valuation; see Remark 5.7.

- **Step 1**. Fix a syntactical notion of "simple" formulas. This usually includes all the literals. Show that all "simple" formulas, except equations in the field, define open sets in (the product of) the valuation topology. This is where the rather special interpretation of P_n in a p-field K is needed in [10], which guarantees that P_n is a clopen subgroup of $K^\times$. Note that P_n is not closed in the valuation topology on K as there is no open neighborhood of 0 that does not intersect with P_n. Also note that, for each formula $\varphi(X)$, that it defines an open set can be expressed by a first-order sentence:

$$\forall X \ (\varphi(X) \to \exists Y \ (v(Y) > v(X) \wedge \forall Z \ (v(Z) > v(Y) \to \varphi(X + Z)))).$$

- **Step 2**. Suppose that a monic polynomial $F(X, \bar{a}) \in \mathcal{O}[X]$ is a counterexample to a version of Hensel's Lemma, where $\bar{a}$ are the (nonzero) coefficients. For example, $F(s, \bar{a}) \in \mathcal{M}$ but $F'(s, \bar{a}) \notin \mathcal{M}$ for some $s \in \mathcal{O}$ and $F(X, \bar{a})$ has no root in K. By assumption, the formula that defines the tuples of the coefficients of all such counterexamples for a fixed degree is equivalent to a formula φ that is quantifier-free (in the main sort) and is in disjunctive normal form. Through some algebraic manipulations it can be shown that one of the disjuncts φ_0 of φ defines a nonempty set $\varphi_0(K^n)$ that is not contained in any proper Zariski closed subset of K^n; that is, φ_0 lacks equational conditions and hence, by Step 1, defines a nonempty open set in K^n. Without loss of generality $\bar{a} \in \varphi_0(K^n)$. For details see [10, Theorem 1, 4].

- **Step 3**. If K is dense in its henselization K^h then the approximation can be carried out as follows: Choose a root $r \in K^h$ of $F(X, \bar{a})$ and write

$$F(X, \bar{a}) = (X + r)F^*(X, \bar{b}),$$

where $\bar{b} \in K^h$ are the (nonzero) coefficients of F^*. Let $U \subseteq \varphi_0(K^n)$ be an open neighborhood of $\bar{a}$, where φ_0 is as in Step 2. Now we can choose $r', \bar{b}' \in K$ that are arbitrarily close to $r, \bar{b}$ with respect to the valuation. Write

$$F(X, \bar{a}') = (X + r')F^*(X, \bar{b}').$$

So $\bar{a}' \in U$, which contradicts the choice of U.

However, in general K is not dense in its henselization. The solution to this in [10] is to consider the field A of algebraic numbers of K. By the assumptions there, in particular that K is a p-field, A cannot be henselian. On the other hand, A has $\mathbb{Z}$ as its value group, which is an ordered abelian group of rank 1 (that is, a subgroup of the additive group of $\mathbb{R}$ with the canonical ordering). It is well-known that if a valuation v for K is of rank 1 then K is dense in its henselization; see the discussion in [6, p. 53].

One may use a more general method to deal with this problem. Using the Omitting Types Theorem, another valued field (L, w) may be constructed such that (L, w) is elementarily equivalent to (K, v) with respect to the language in question and w is of rank 1. For example, this method is used in [3] to obtain a converse QE result for real closed valuation rings. We will also use it below to establish a few converse QE results.

Note that Step 2 can always be implemented for any valued field that is not henselian. So the bulk of the work in the sequel will concentrate on Step 1 and Step 3.

Next we will describe languages of a quite different kind, namely the Denef-Pas style languages.

DEFINITION 2.2. Let K be a valued field and $\overline{K}$ its residue field. An *angular component map* is a function $\overline{\mathrm{ac}} : K \longrightarrow \overline{K}$ such that

1. $\overline{\mathrm{ac}}\, 0 = 0$,
2. the restriction $\overline{\mathrm{ac}} \upharpoonright K^{\times}$ is a group homomorphism $K^{\times} \longrightarrow \overline{K}^{\times}$,
3. the restriction $\overline{\mathrm{ac}} \upharpoonright (\mathcal{O} \setminus \mathcal{M})$ is the projection map, that is, $\overline{\mathrm{ac}}\, u = u + \mathcal{M}$ for all $u \in \mathcal{O} \setminus \mathcal{M}$.

The template of Denef-Pas style languages has three sorts: the field sort which is the main sort, the residue field sort, and the value group sort. These are usually denoted by K, $\overline{K}$, and Γ. The K-sort and $\overline{K}$-sort use the language $\mathcal{L}_R$ of rings. The Γ-sort uses the langauge $\mathcal{L}_{OG}$ of ordered groups, $\{+, <, 0\}$, and an additional symbol ∞ that designates the top element in the ordering. There are two cross-sort function symbols: $v : K \longrightarrow \Gamma$, which stands for the valuation, and $\overline{\mathrm{ac}} : K \longrightarrow \overline{K}$, which stands for an angular component map.

Any language that expands this template is a Denef-Pas language. A prototypical example is the language $\mathcal{L}_{RRP}$ used in [11], in which the field sort and the residue field sort use the language $\mathcal{L}_R$ and the Γ-sort uses the language $\mathcal{L}_{Pr\infty} = \mathcal{L}_{Pr} \cup \{\infty\}$, where $\mathcal{L}_{Pr}$ is the Presburger language $\{+, -, <, 0, 1\} \cup \{D_n : n > 1\}$. Let $S = \langle K, \overline{K}, \Gamma \cup \{\infty\}, v, \overline{\mathrm{ac}} \rangle$ be a structure of $\mathcal{L}_{RRP}$. One of the main results of [11] is that if K is henselian and both K and $\overline{K}$ are of characteristic 0 then $\mathrm{Th}(S)$ admits QE in the K-sort; that is, for every formula φ in $\mathcal{L}_{RRP}$ there is a formula φ^* in $\mathcal{L}_{RRP}$ that does not contain K-quantifiers such that $S \models \varphi \leftrightarrow \varphi^*$. A converse of this with respect to henselianity will be established in Section 5.

The following notions are formulated for any Denef-Pas language $\mathcal{L}$, where we use $\mathcal{L}_K$, $\mathcal{L}_{\overline{K}}$, and $\mathcal{L}_{\Gamma\infty}$ to denote the languages used by the three sorts.

DEFINITION 2.3. A formula φ in $\mathcal{L}$ is *simple* if φ does not contain any K-quantifiers.

DEFINITION 2.4. A formula φ in $\mathcal{L}_K \cup \mathcal{L}_{\Gamma\infty}$ is a Γ-*formula* if it does not contain K-quantifiers and atomic formulas in $\mathcal{L}_K$. Similarly a formula φ in $\mathcal{L}_K \cup \mathcal{L}_{\overline{K}}$ is a $\overline{K}$-*formula* if it does not contain K-quantifiers and atomic formulas in $\mathcal{L}_K$.

§3. Henselianity and the Macintyre language. In this section we shall describe some conditions under which any valued field that admits QE in the Macintyre language $\mathcal{L}_{Mac}$ is henselian. The bulk of the work will concentrate on the density condition in Step 3. To satisfy that one can certainly impose some Galois theoretic conditions on K that guarantees that K is dense in its henselization; see [5, Theorem 2.15]. However this does not seem to be very satisfactory either as it does not bear much on the intrinsic algebraic structure of the valued field in question. Below more elementary conditions will be given. An obvious advantage of this approach is that one can easily construct such

valued fields. We assume that the reader is familiar with the basics of the theory of valued fields. A good source for this is [6].

There will be different conditions depending on whether the residue characteristic is zero. But first we shall describe some concepts that are used in these conditions. Let (L, w) be a valued field. Let $\mathcal{O}$ be the valuation ring and $\mathcal{M}$ its maximal ideal.

For $r, t \in \mathcal{O}$ we say that they are *comparable*, written as $r \asymp t$, if there is a natural number n such that either $w(r^n) \leq w(t) \leq w(r^{n+1})$ or $w(t^n) \leq w(r) \leq w(t^{n+1})$. They are *incomparable* if they are not comparable. We write $r \ll t$ if r, t are incomparable and $w(r) < w(t)$. If $t \in A \subseteq L$ and the set $\{nw(t) : n \in \mathbb{N}\}$ is cofinal in the set $\{w(r) : r \in A\}$ then we say that t is a *cofinal element* in A. Note that for all units $r \in \mathcal{O} \setminus \mathcal{M}$ and all $s \in \mathcal{M}$ we have $r \ll t$. Obviously $r \ll 0$ for any nonzero $r \in \mathcal{O}$. For $t \in \mathcal{M}$ we write $\mathrm{char}(\overline{L}) \ll t$ if either $\mathrm{char}(\overline{L}) = 0$ or $\mathrm{char}(\overline{L}) = p > 0$ and $p \ll t$. If $r \ll t$ for every $r \in A \subseteq \mathcal{O}$ then we simply write $A \ll t$. Similarly we write $A \asymp t$ if there is an $r \in A$ such that $r \asymp t$ and r is a cofinal element in A.

If R is a subring of a field L then we write R^L for the integral closure of R in L. For any $A \subseteq L$ we write $\mathbb{Q}(A)$ for the smallest subfield generated by A in L. Note that $\mathbb{Q}(A)^L$ is the algebraic closure of $\mathbb{Q}(A)$ in L and $(\mathbb{Q}(A) \cap \mathcal{O})^L \subseteq \mathbb{Q}(A)^L \cap \mathcal{O}$.

DEFINITION 3.1. We say that (L, w) is of *prohenselian degree* n if for any natural number $1 \leq m \leq n$ the valuation ring $\mathbb{Q}(t_1, \ldots, t_m)^L \cap \mathcal{O}$ admits a henselian coarsening for every sequence $t_1, \ldots, t_m \in \mathcal{M}$ with $t_m \gg \ldots \gg t_1$. If (L, w) is of prohenselian degree n for every natural number n then it is *prohenselian*.

When does a valuation admit a henselian coarsening? One answer, Corollary 3.6, is this: If it lives near a henselian valuation and is not *antihenselian*:

PROPOSITION 3.2. *Let L^h be the henselization of (L, w). The following are equivalent:*

1. *L^h is the separable closure of L.*
2. *If L^* is a finite separable extension of L then w has $[L^* : L]$ distinct prolongations in L^*.*
3. *The valuation w is saturated (i.e. wL is divisible and $\overline{L}$ is algebraically closed) and defectless.*

If any one of the three conditions is satisfied then w is called an antihenselian *valuation.*

PROOF. This is well-known; see, for example, the first section of [4]. A proof can be quite easily assembled from various results in [6, Section 5]. For example, if $\overline{L}$ is algebraically closed then the inertia group equals to the decomposition group and if wL is divisible then the ramification group equals to the inertia group, hence the inertia field and the ramification field all equal to L^h. Since w is defectless, the ramification field is the separable closure of L. ⊣

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two valuation rings of the field L. we say that $\mathcal{O}_1$ and $\mathcal{O}_2$ are *dependent* if the smallest subring $\mathcal{O}_1 \mathcal{O}_2$ of L that contains both $\mathcal{O}_1$ and $\mathcal{O}_2$ is a proper subring of L.

THEOREM 3.3 (F. K. Schmidt). *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two henselian valuation rings of the field L. If L is not separably closed, then $\mathcal{O}_1$ and $\mathcal{O}_2$ are dependent.*

PROOF. See [6, Theorem 4.4.1].   ⊣

PROPOSITION 3.4. *Suppose that L^*/L is an algebraic extension of fields, $\mathcal{O}$ is a valuation ring of L, and $\mathcal{O}_1, \mathcal{O}_2$ are two prolongations of $\mathcal{O}$ in L^*. If $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then $\mathcal{O}_1 = \mathcal{O}_2$.*

PROOF. See [6, Lemma 3.2.8].   ⊣

THEOREM 3.5. *Let $\mathcal{O} \subseteq \mathcal{O}_1$ be two valuation rings of L with corresponding maximal ideals $\mathcal{M}_1 \subseteq \mathcal{M}$. Then $\overline{\mathcal{O}} = \mathcal{O}/\mathcal{M}_1$ is a valuation ring of the field $\overline{L} = \mathcal{O}_1/\mathcal{M}_1$. The composition $(L, \mathcal{O})$ is henselian iff both $(L, \mathcal{O}_1)$ and $(\overline{L}, \overline{\mathcal{O}})$ are henselian.*

PROOF. See [6, Corollary 4.1.4].   ⊣

From these facts we easily deduce:

COROLLARY 3.6. *Let $\mathcal{O}$ be a henselian valuation ring of L. Then for every non-antihenselian valuation ring $\mathcal{O}_1$ of L there is a henselian coarsening $\hat{\mathcal{O}}_1$ of $\mathcal{O}_1$.*

PROOF. If $\mathcal{O}_1$ is henselian then we are done. So assume that $\mathcal{O}_1$ is not henselian. Since $\mathcal{O}_1$ is not antihenselian, the henselization $L^h(\mathcal{O}_1)$ of L with respect to $\mathcal{O}_1$ is not separably closed. Let $\mathcal{O}_1^h$ be a henselian prolongation of $\mathcal{O}_1$ in $L^h(\mathcal{O}_1)$. Note that such a prolongation may not be unique. Since $\mathcal{O}$ is henselian, the unique prolongation $\mathcal{O}'$ of $\mathcal{O}$ in $L^h(\mathcal{O}_1)$ is also henselian. So there is a valuation ring $\mathcal{O}_2'$ of $L^h(\mathcal{O}_1)$ that contains both $\mathcal{O}'$ and $\mathcal{O}_1^h$. By Theorem 3.5 $\mathcal{O}_2'$ is henselian. Let $\mathcal{O}_2 = \mathcal{O}_2' \cap L$. By Proposition 3.4 $\mathcal{O}_2$ is a proper subring of L. Since $\mathcal{O} \subseteq \mathcal{O}_2$, $\mathcal{O}_2$ is henselian by Theorem 3.5 again and contains $\mathcal{O}_1$, as desired.   ⊣

Remark 3.7. That a field carries a henselian valuation is not a first-order property in the language $\mathcal{L}_R$. Consider the example in [13, p. 338]. There an inverse limit L of valued fields is constructed such that

- L is neither algebraically closed nor real closed,
- L is elementarily equivalent to a henselian valued field with respect to $\mathcal{L}_R$,
- no valuation of L is henselian.

A subgroup H of an ordered abelian group G is *convex* if, for every $a \in G$, $0 \le a \le b$ for some $b \in H$ implies $a \in H$. Obviously the set of all convex subgroups of G are linearly ordered by inclusion. The order type of this set is called the *rank* of G, denoted by $\mathrm{rk}\, G$. If $\mathrm{rk}\, G$ is finite then we identify it with a natural number. For example, $\mathrm{rk}\, G = 0$ if and only if $G = \{0\}$. Groups of rank 1, that is, groups with only one proper convex subgroup $\{0\}$, are of particular importance for Step 3 in Section 2, because of the following well-known fact:

FACT 3.8. *Let (L, w) be a valued field. If the value group wL is of rank 1 then L is dense in the henselization L^h (with respect to the valuation topology).*

The following characterization of ordered abelian groups of rank 1 has already been mentioned in passing above:

PROPOSITION 3.9. *A group G is of rank 1 if and only if it is order-isomorphic to a non-trivial subgroup of the (canonically) ordered additive subgroup of the reals.*

PROOF. See [6, Proposition 2.1.1]. ⊣

For the rest of this section let (K, v) be a valued field and $\mathcal{O}_v, \mathcal{M}_v$ its valuation ring and maximal ideal, respectively.

3.1. The residue characteristic is zero. Throughout this subsection we assume that $\mathrm{char}(\overline{K}) = 0$, $\mathrm{Th}(K)$ admits QE in $\mathcal{L}_{Mac}$, and (K, v) is of pro-henselian degree 2. We shall first consider (K, v) as a structure of $\mathcal{L}_{Mac}$ where, unlike in p-fields, each predicate P_n is interpreted naturally as the subgroup of nth powers of $K^\times$. We do not assume that K satisfies these other defining conditions for a p-adically closed field because they are immaterial to the discussion below. We shall prove:

THEOREM 3.10. *Under these conditions, the valuation v is henselian.*

Step 1 in Section 2 can be carried out easily for (K, v).

LEMMA 3.11. *For every $n > 1$ the subgroup P_n of $K^\times$ is clopen in the valuation topology induced by v.*

PROOF. For every $t \in \mathcal{M}_v$, t is clearly a cofinal element in $\mathbb{Q}(t)^K$. The restriction of v to $\mathbb{Q}(t)^K$ admits a henselian coarsening, which must be v itself as $\mathrm{rk}\, v\mathbb{Q}(t)^K = 1$. Since $v(t) > v(n) = 0$ for every $n > 1$. So by Hensel's Lemma $1 + t$ is an nth power in $\mathbb{Q}(t)^K$, hence in K. So P_n contains an open neighborhood of 1 in K and hence is open in the valuation topology induced by v. It is also closed as it is a subgroup of $K^\times$. ⊣

Next, note that the relation $v(X) \leq v(Y)$ is not quantifier-free definable in $\mathcal{L}_{Mac}$. See the discussion in [10, p. 82]. However, since the relation is definable in $\mathcal{L}_{Mac}$, we shall use it as a shorthand for the corresponding formula in $\mathcal{L}_{Mac}$. To carry out Step 3 we shall apply the Omitting Types Theorem to achieve the density condition. Our goal is to show that the following 2-type

(3.1) $$\Phi(X, Y) = \{0 < v(X^n) < v(Y) \wedge Y \neq 0 : n \geq 1\}$$

is not isolated modulo $\mathrm{Th}(K)$. To that end, we suppose for contradiction that there is a formula $\pi(X, Y)$ in $\mathcal{L}_{Mac}$ such that

- $\exists X, Y\ \pi(X, Y) \in \mathrm{Th}(K)$ and
- $\pi(X, Y) \vdash \Phi(X, Y)$ modulo $\mathrm{Th}(K)$.

Let $r, t \in \mathcal{M}_v$ such that $r \ll t$ and $K \models \pi(r, t)$. Since K admits QE in $\mathcal{L}_{Mac}$, without loss of generality we may assume that $\pi(X, Y)$ is of the form

(3.2) $$\bigwedge_i E_i(X, Y) = 0 \wedge F(X, Y) \neq 0 \wedge \bigwedge_k \mathcal{O}(R_k(X, Y))$$
$$\wedge \bigwedge_m P_{u_m}(T_m(X, Y)) \wedge \bigwedge_n \neg P_{u_n}(U_n(X, Y)),$$

where $E_i, F, R_k, T_m, U_n \in \mathbb{Z}[X, Y]$. Note that $\pi(X, Y)$ does not contain literals of the form $\neg \mathcal{O}(S(X, Y))$ with $S \in \mathbb{Z}[X, Y]$.

The following lemma shows that in fact $\pi(X, Y)$ does not contain equations.

LEMMA 3.12. *For any nonzero polynomial* $F(X,Y) \in \mathbb{Z}[X,Y]$, $F(r,t) \neq 0$.

PROOF. Suppose for contradiction $F(r,t) = 0$. Write $F(X,Y)$ as

$$(3.3) \qquad\qquad F_n(X)Y^n + \ldots + F_0(X),$$

where $F_0(X), \ldots, F_n(X) \in \mathbb{Z}[X]$ are not all zero. If $F(X,Y)$ is a monomial in Y then it can be written as

$$(3.4) \qquad\qquad (e_m X^m + \ldots + e_0)Y^i$$

for some $0 \leq i \leq n$, where $e_0, \ldots, e_m \in \mathbb{Z}[X]$ are not all zero. But no two summands in $e_m r^m + \ldots + e_0$ have the same valuation, for otherwise we would have $v(r) = 0$. Hence $v(e_m r^m + \ldots + e_0) < \infty$, contradiction.

So we may assume that $F(X,Y)$ has at least two nonzero monomial summands. Now for some $i > j \geq 0$ we have $v(F_i(r)t^i) = v(F_j(r)t^j)$. So

$$v(t^{i-j}) = v(F_j(r)/F_i(r)).$$

But again, in each $F_k(r)$, no two summands have the same valuation, so $F_k(r) \ll t$. So $t \asymp r$ at the largest, contradiction again.   ⊣

Now, the formula $\pi(X,Y)$ can actually be satisfied by elements in K that are comparable.

LEMMA 3.13. *Suppose that* $K \models P_u(E(r,t))$, *where* $E(X,Y) \in \mathbb{Z}[X,Y]$ *are nonzero. Then for sufficiently large natural number* k

$$K \models P_u(E(rt^u, t^{ku+1})).$$

PROOF. Fix a natural number k. Write $E(X,Y)$ as

$$E_0(X)Y^e \left(\frac{E_n(X)}{E_0(X)}Y^n + \ldots + \frac{E_1(X)}{E_0(X)}Y + 1 \right)$$

with $E_0(X), E_n(X) \in \mathbb{Z}[X]$ nonzero. Write $E_0(X)$ as

$$a_0 X^d \left(\frac{a_m}{a_0}X^m + \ldots + \frac{a_1}{a_0}X + 1 \right)$$

with $a_0, a_m \in \mathbb{Z}$ nonzero.

Let $\hat{v}$ be a henselian coarsening of the restriction of v to $\mathbb{Q}(r,t)^K$. Since clearly $\hat{v}(t) > 0$, we see that actually

$$(3.5) \qquad \mathbb{Q}(r,t)^K \models P_u \left(\frac{E_n(r)}{E_0(r)}t^n + \ldots + \frac{E_1(r)}{E_0(r)}t + 1 \right).$$

Similarly we get

$$\mathbb{Q}(r)^K \models P_u \left(\frac{a_m}{a_0}r^m + \ldots + \frac{a_1}{a_0}r + 1 \right).$$

So we must have

$$(3.6) \qquad\qquad K \models P_u(a_0 r^d t^e).$$

Substituting rt^u, t^{ku+1} for r, t respectively we see that

$$K \models P_u(a_0 r^d t^{ud} t^{kue} t^e).$$

Applying Hensel's Lemma in $\mathbb{Q}(r,t)^K$ when k is sufficiently large we deduce that $K \models P_u(E(rt^u, t^{ku+1}))$.   ⊣

LEMMA 3.14. *Let $u = \prod_m u_m \prod_n u_n$. For sufficiently large natural number k, $K \models \pi(rt^u, t^{ku+1})$. Hence $\pi(X, Y)$ cannot isolate the type $\Phi(X, Y)$ modulo $\mathrm{Th}(K)$.*

PROOF. We have seen that $\pi(X, Y)$ does not contain equations. Also, if k is sufficiently large then clearly the inequality in $\pi(X, Y)$ is satisfied by rt^u, t^{ku+1}. Hence it remains to show that for infinitely many k

$$K \models \bigwedge_m P_{u_m}(T_m(rt^u, t^{ku+1})) \wedge \bigwedge_n \neg P_{u_n}(U_n(rt^u, t^{ku+1})).$$

Now with the current choice of u and a sufficiently large k clearly the argument for the last lemma works for each u_m. On the other hand, if we run that argument for $\neg P_{u_n}(U_n(r, t))$ then (3.6) turns into

$$K \models \neg P_{u_n}(a_0 r^d t^e).$$

So it is easy to see that if k is sufficiently large then $K \models \neg P_{u_n}(U_n(rt^u, t^{ku+1}))$ for each n. ⊣

THEOREM 3.15. *There is a valued field (L, w) such that w is of rank 1 and $(L, w) \equiv (K, v)$ as structures of $\mathcal{L}_{Mac}$.*

PROOF. Immediate by the Omitting Types Theorem and the last lemma. ⊣

This shows that Step 3 in Section 2 can be carried out for (K, v).

3.2. The residue characteristic is nonzero. Throughout this subsection we assume that $\mathrm{char}(\overline{K}) = p > 0$, $\mathrm{Th}(K)$ admits QE in $\mathcal{L}_{Mac}$, and (K, v) is of prohenselian degree 1. We also assume that (K, v) is *tight*; that is, $v(p)$ is contained in the smallest nonzero convex subgroup of vK. There is still one more condition for (K, v).

DEFINITION 3.16. Let $\mathcal{C}$ be a subgroup of $K^\times$ such that $\mathbb{Q}^\times \subseteq \mathcal{C}$. We say that $\mathcal{C}$ is *conservative* if

1. p is a cofinal element in $\mathbb{Q}(r)^K$ for every $r \in \mathcal{C}$,
2. $\mathcal{C}$ is an existentially closed substructure of $K^\times$ over $\mathbb{Q}^\times$ (that is, with parameters in $\mathbb{Q}^\times$) with respect to the language $\mathcal{L}_G$ of groups.

Let (L, w) be a tight valued field with $\mathrm{char}(\overline{L}) = p > 0$ and A the subfield of algebraic numbers of L. Clearly $\mathrm{rk}\, wA = 1$. If (L, w) is a p-adic closed field of p-rank 1 (or of any p-rank), then (A, w) is a p-adic closed field of p-rank 1 and, by Macintyre's Theorem, (A, w) is an elementary substructure of (L, w) with respect to $\mathcal{L}_{Mac}$. So $A^\times$ is a conservative subgroup of $L^\times$. Another obvious example is when A is a pseudo algebraically closed field (PAC field), since a field is PAC if and only if it is existentially closed in every regular extension (with respect to $\mathcal{L}_R$, of course). Such valued fields are abundant since every algebraic extension of a PAC field is PAC. For these and other basic facts about PAC fields see [7, Chapter 11].

Fix a natural number n. Suppose that $A^\times$ is a conservative subgroup of $L^\times$ and w is a henselian valuation with $\mathrm{rk}\, wL > 1$. Now it is actually easy to construct a valued field (K, v) such that

- $\mathrm{char}(\overline{K}) = p > 0$ and (K, v) is tight,

- (K, v) is of prohenselian degree n,
- there is a conservative subgroup $\mathcal{C}$ of $K^\times$.

We start with a subgroup $\mathcal{C}$ of $A^\times$ such that $\mathbb{Q}^\times \subseteq \mathcal{C}$ and $\mathcal{C}$ is an existentially closed $\mathcal{L}_G$-substructure of $A^\times$ (hence of $L^\times$) over $\mathbb{Q}^\times$. Pick an element $t \in L$ with $t \gg p$ and let K_0 be a subfield of L such that $\mathcal{C} \cup \{t\} \subseteq K_0$. Of course the induced valued field (K_0, w) may fail to be of prehenselian degree n. However, since prohenseliality is a sort of "closure" condition for partial henselianity and (L, w) is henselian, we can simply find a subfield K_1 of L such that

- $K_0 \subseteq K_1$,
- $\mathbb{Q}(t_1, \dots, t_n)^L \subseteq K_1$ for any $t_1, \dots, t_n \in \mathcal{M} \cap K_0$ with $t_n \gg \dots \gg t_1$.

Then we proceed to find a subfield K_2 of L that satisfies the above two conditions with respect to K_1. In this fashion we can construct a sequence of subfields $K_0, \dots, K_i, \dots$ of L such that $K = \bigcup_i K_i$ is as desired, where the conservative subgroup in question is $\mathcal{C}$.

For the rest of this subsection we assume that there is a conservative subgroup $\mathcal{C}$ of $K^\times$.

LEMMA 3.17. *For every $n > 1$ the subgroup P_n of $K^\times$ is clopen in the valuation topology induced by v.*

PROOF. If there is a $t \in \mathcal{M}_v$ with $p \ll t$ then we may simply repeat the argument in Lemma 3.11. If $\mathrm{char}(\overline{K}) = p > 0$ is a cofinal element in K, then for any $n > 1$ we consider any $t \in \mathcal{M}_v$ with $v(t) > 2v(n)$. Since $\mathrm{rk}\, vK = 1$ and (K, v) is of prohenselian degree 1, the restriction of v to $\mathbb{Q}(t)^K$ is henselian. So by Newton's Lemma $1 + t$ is an nth power in $\mathbb{Q}(t)^K$, hence in K.   ⊣

LEMMA 3.18. *Let $E_i, F_j \in \mathbb{Z}[X]$ and $x \in \mathcal{M}_v$ with $x \gg p$ such that*

$$K \models \bigwedge_i P_{u_i}(E_i(x)) \wedge \bigwedge_j \neg P_{u_j}(F_j(x)).$$

Let $u = \prod_i u_i \prod_j u_j$. Then for some x^* with $x^* \asymp p$

$$K \models \bigwedge_i P_{u_i}(E_i(x^*)) \wedge \bigwedge_j \neg P_{u_j}(F_j(x^*)).$$

PROOF. Let us begin by considering just one polynomial, say, $E_1(X)$. Write it as

(3.7) $$a_0 X^m \left(\frac{a_n}{a_0} X^n + \dots + \frac{a_1}{a_0} X + 1 \right),$$

where $a_n, \dots, a_0 \in \mathbb{Z}$, $a_n, a_0 \neq 0$, and $n \geq m \geq 0$. Let $\hat{v}$ be a henselian coarsening of the restriction of v to $\mathbb{Q}(x)^K$. Since x is clearly a cofinal element in $\mathbb{Q}(x)^K$, we may assume that $\hat{v}\mathbb{Q} = 0$. By Hensel's Lemma we see that

$$\mathbb{Q}(x)^K \models P_{u_1}\left(\frac{a_n}{a_0} x^n + \dots + \frac{a_1}{a_0} x + 1 \right).$$

So we have

(3.8) $$K \models P_{u_1}(a_0 x^m).$$

It is easy to see that the above argument does not depend on the number of polynomials under consideration. On the other hand, if we run the argument for $\neg P_{u_j}(F_j(X))$ then (3.8) turns into

$$K \models \neg P_{u_j}(a_0 x^m),$$

for each j. So we have

$$K \models \bigwedge_i P_{u_i}(a_i x^{m_i}) \wedge \bigwedge_j \neg P_{u_j}(a_j x^{m_j})$$

for some $a_i, a_j, m_i, m_j \in \mathbb{Z}$. Since $\mathcal{C}$ is an elementary $\mathcal{L}_G$-substructure of $K^\times$, there is an $x_* \in \mathcal{C}$ such that

$$K \models \bigwedge_i P_{u_i}(a_i x_*^{m_i}) \wedge \bigwedge_j \neg P_{u_j}(a_j x_*^{m_j}).$$

Since p is a cofinal element in $\mathbb{Q}(x_*)^K$, we have $\operatorname{rk}\mathbb{Q}(x_*)^K = 1$ and the restriction of v to $\mathbb{Q}(x_*)^K$ is henselian. So by Newton's Lemma, for sufficiently large natural number k,

$$\mathbb{Q}(x_*)^K \models P_{u_i}\left(\frac{a_n}{a_0}(p^{ku} x_*)^n + \ldots + \frac{a_1}{a_0} p^{ku} x_* + 1 \right).$$

for each u_i, and similarly for each u_j. So $x^* = p^{ku} x_*$ for sufficiently large k is as desired. ⊣

Again we use the Omitting Types Theorem to show that Step 3 can be carried out.

THEOREM 3.19. *There is a valued field (L, w) such that w is of rank 1 and $(L, w) \equiv (K, v)$ as structures of $\mathcal{L}_{Mac}$.*

PROOF. It suffices to omit the 2-type (3.1). Suppose for contradiction it is not omitted. Let $\pi(X, Y)$ be as in the last subsection and $r, t \in \mathcal{M}_v$ such that $r \ll t$ and $K \models \pi(r, t)$. Since (K, v) is tight, clearly $p \ll t$. Consider the existential formula $\exists X\ \pi(X, Y)$. Since K admits QE in $\mathcal{L}_{Mac}$, there is a quantifier-free formula $\bigvee_i \varphi_i(Y)$ in disjunctive normal form such that

$$K \models \exists X\ \pi(X, Y) \leftrightarrow \bigvee_i \varphi_i(Y).$$

Without loss of generality $K \models \varphi_1(t)$. Then the proof of Lemma 3.12 shows that $\varphi_1(Y)$ does not contain equations. By Lemma 3.18 there is a t^* with $t^* \asymp p$ that satisfies all the literals that involve nth power predicates in $\varphi_1(Y)$. It is also clear from the proof there that t^* may be chosen so that the inequality in $\varphi_1(Y)$ is also satisfied by t^*. This is a contradiction since (K, v) is tight. ⊣

3.3. A variation of the Macintyre language. There is a quite useful variation $\mathcal{L}_{Mac,D}$ of the Macintyre language which uses a function instead of a predicate for the valuation ring. Let (L, w) be a valued field. Define a restricted division function $D : L^2 \longrightarrow L^2$ by

$$(x, y) \longmapsto \begin{cases} x/y, & \text{if } w(x) \geq w(y) \text{ and } y \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

The behavior of D can be axiomatized; see the definition in [10, p. 82], where the binary predicate "X div Y" can be expressed as a quantifier-free formula $Y = 0 \vee D(Y, X) \neq 0$. So the language $\mathcal{L}_{Mac,D}$ is more expressive than the language $\mathcal{L}_{Mac}$.

LEMMA 3.20. *Let $\overline{X}$ be a tuple of variables. Every conjunction $\varphi(\overline{X})$ of literals in $\mathcal{L}_{Mac,D}$ is equivalent to a disjunction of formulas of the form:*

$$(3.9) \quad \bigwedge_j \exists Y_j \ (Y_j \neq 0 \wedge Y_j G_j(\overline{X}) = H_j(\overline{X}) \wedge D(Y_j, 1) = Y_j)$$

$$\wedge \bigwedge_i E_i(\overline{X}) = 0 \wedge F(\overline{X}) \neq 0 \wedge \bigwedge_m P_{u_m}(D(T_m(\overline{X}), S_m(\overline{X})))$$

$$\wedge \bigwedge_n \neg P_{u_n}(D(U_n(\overline{X}), V_n(\overline{X}))),$$

where $G_j, H_j, E_i, F, T_m, S_m, U_n, V_n \in \mathbb{Z}[\overline{X}]$.

PROOF. Since the function D behaves as division whenever its output is not 0, the claim essentially says that the "denominators" in the terms can be cleared when the defining conditions for the occurrences of D are explicitly stated. For example, if $E(X), F(X) \in \mathbb{Z}[X]$, then $D(E(X), F(X)) = 0$ is equivalent to

$$\exists Y \ (Y \neq 0 \wedge Y E(X) = F(X) \wedge D(Y, 1) = Y) \vee F(X) = 0 \vee E(X) = 0.$$

It is not hard to see that the claim follows from a routine induction on how deeply the symbol D is nested in φ. $\dashv$ $\dashv$

Under the same conditions, the results in the last two subsections also hold with respect to $\mathcal{L}_{Mac,D}$.

THEOREM 3.21. *Suppose that $\mathrm{Th}(K)$ admits QE in $\mathcal{L}_{Mac,D}$ and*

- *if $\mathrm{char}(\overline{K}) = 0$ then (K, v) is of prohenselian degree 2;*
- *if $\mathrm{char}(\overline{K}) = p > 0$ then (K, v) is of prohenselian degree 1, (K, v) is tight, and there is a conservative subgroup $\mathcal{C}$ of $K^\times$.*

Then the valuation v is henselian.

PROOF. We shall check the three steps in Section 2.

For Step 1 we need to show that, except the equations, all conjuncts in the form (3.9) define open sets. By Lemma 3.11 and Lemma 3.17 each nth power predicate defines a clopen set. Since quotients of polynomials are continuous maps, except the equations all the literal conjuncts in (3.9) define open sets. For the same reason all the existential conjuncts there define open sets.

As before Step 2 can be carried out in exactly the same way. So we may find an open set of coefficients that all witness the failure of henselianity.

For Step 3 we need to show that no formula $\pi(X, Y)$ in $\mathcal{L}_{Mac,D}$ of the form (3.9) can isolate the 2-type (3.1). Suppose for contradiction that there is such a formula $\pi(X, Y)$. Let $r, t \in \mathcal{M}_v$ such that $r \ll t$ and $K \models \pi(r, t)$.

Suppose that $\mathrm{char}(\overline{K}) = 0$. By Lemma 3.12 $\pi(X, Y)$ does not contain equations. Next, if we run the argument of Lemma 3.13 for any conjunct

$$P_{u_m}(D(T_m(X, Y), S_m(X, Y)))$$

of $\pi(X, Y)$, where $T_m(r, t), S_m(r, t) \neq 0$, then (3.6) turns into something of the form

$$K \models P_{u_m}(D(a_0 r^d t^e, b_0 r^f t^g)).$$

So for sufficiently large natural number k

$$K \models P_{u_m}(D(T_m(rt^{u_m}, t^{ku_m+1}), S_m(rt^{u_m}, t^{ku_m+1}))).$$

Similarly we can conclude that, for sufficiently large k, the pair rt^u, t^{ku+1} satisfies every conjunct in $\pi(X, Y)$ except the existential ones, where u is as in Lemma 3.14. For the existential conjuncts, since $\mathbb{Q} \ll r \ll t$, we have

$$v(ar^d t^e) = v(G_j(r, t)) \leq v(H_j(r, t)) = v(br^f t^g)$$

for some natural numbers a, b, d, e, f, g. So either $e < g$ or $e = g$ and $d < f$ or $e = g$, $d = f$, and $v(a) \leq v(b)$. So we see that for sufficiently large k

$$v(G_j(rt^u, t^{ku+1})) = v(a(rt^u)^d(t^{ku+1})^e) \leq v(b(rt^u)^f(t^{ku+1})^g) = v(H_j(rt^u, t^{ku+1})).$$

So indeed we can find a sufficiently large k such that the pair of comparable elements rt^u, t^{ku+1} satisfies every conjunct in $\pi(X, Y)$, which yields a contradiction.

Suppose that $\operatorname{char}(\overline{K}) = p > 0$. So $p \ll t$. There is a formula $\bigvee_i \varphi_i(Y)$ such that each $\varphi_i(Y)$ is in the form (3.9) and

$$K \models \exists X\ \pi(X, Y) \leftrightarrow \bigvee_i \varphi_i(Y).$$

Say, $K \models \varphi_1(t)$. Then the proof of Lemma 3.12 shows that $\varphi_1(Y)$ does not contain equations. Modifying the proof of Lemma 3.18 as in the last paragraph we see that there is a t^* with $t^* \asymp p$ that satisfies all the literals that involve nth power predicates and the inequality in $\varphi_1(Y)$. Moreover for any n this t^* may be chosen so that $v(t^*) > v(p^n)$. Now, since $t \gg p$, for each existential conjunct in $\varphi_1(Y)$ we have

$$v(at^e) = v(G_j(t)) \leq v(H_j(t)) = v(bt^g)$$

for some natural numbers a, b, e, g with either $e < g$ or $e = g$ and $v(a) \leq v(b)$. So t^* may be chosen so that

$$v(G_j(t^*)) = v(a(t^*)^e) \leq v(b(t^*)^g) = v(H_j(t^*)).$$

So there is a t^* with $t^* \asymp p$ such that $K \models \exists X\ \pi(X, t^*)$. This is a contradiction since (K, v) is tight. $\dashv$

§4. **Henselianity without QE.** In the last section we have seen that if a valued field is of bounded prohenselian degree then QE and other logical conditions are needed to show henselianity. In this short section we shall see that prohenselianity may imply henselianity without logical conditions.

Let L be a field of finite transcendence degree and w a valuation of L. Let $\mathcal{O}, \mathcal{M}$ be its valuation ring and maximal ideal. For any extension of fields L/K we write $\operatorname{tr\,deg} L/K$ for the transcendence degree of L over K. If K is the prime field of L then we simply write $\operatorname{tr\,deg} L$. We shall need the following fact:

PROPOSITION 4.1. *Suppose that L/K is a field extension and both* $\operatorname{rk} wL$ *and* $\operatorname{tr} \deg L/K$ *are finite. Then*

$$\operatorname{tr} \deg \overline{L}/\overline{K} + \operatorname{rk} wL \le \operatorname{tr} \deg L/K + \operatorname{rk} wK.$$

PROOF. See [6, Corollary 3.4.4].   ⊣

By the proof of Lemma 3.12 we have $\operatorname{rk} wL \le \operatorname{tr} \deg L + 1$. We say that (L, w) is *flat* if

1. $\operatorname{rk} wL \ge \operatorname{tr} \deg L$;
2. if $\operatorname{char}(\overline{L}) = p > 0$ then (L, w) is tight and if moreover $\operatorname{rk} wL = \operatorname{tr} \deg L$ then there is a transcendental element t such that $\operatorname{rk} w\mathbb{Q}(t)^L = 1$.

Now, if L carries a henselian valuation then it cannot be an finite extension of $K(t)$, where K is a subfield of L and t is transcendental over K; see [8, Proposition 21]. Moreover, if every subfield $K^L \subseteq L$ carries a henselian valuation and (K^L, w) is not antihenselian then by Corollary 3.6 (L, w) is prohenselian. It is not hard to block antihenseliality for each (K^L, w). For example, if $\operatorname{char}(\overline{L}) = p > 0$ and $w(p)$ is not divisible in wL then, for all subfield $K \subseteq L$, wK^L is not divisible and hence by Proposition 3.2 (K^L, w) is not antihenselian. Of course if $\operatorname{rk} wL = \operatorname{tr} \deg L \le 1$ then prohenseliality and henseliality are the same. So the following proposition is really about valued fields whose value groups have high ranks.

PROPOSITION 4.2. *If (L, w) is flat and prohenselian then w is a henselian valuation.*

PROOF. The proof is by induction on $\operatorname{tr} \deg L$. For the base case we have $\operatorname{tr} \deg L = 1$ if $\operatorname{char}(\overline{L}) = 0$ or $\operatorname{tr} \deg L \le 1$ if $\operatorname{char}(\overline{L}) = p > 0$ and $\operatorname{rk} wL = 1$. Since in both cases w cannot be coarsened as $\operatorname{rk} wL = 1$, it must be henselian.

Now suppose that $\operatorname{tr} \deg L = n + 1$. Since (L, w) is flat, we may choose a transcendence base $\{t_1, \dots, t_{n+1}\} \subseteq \mathcal{M}$ of L such that

- $\operatorname{char}(\overline{L}) \ll t_1 \ll \dots \ll t_{n+1}$ or $\operatorname{char}(\overline{L}) \asymp t_1 \ll \dots \ll t_{n+1}$ if $\operatorname{char}(\overline{L}) = p > 0$ and $\operatorname{rk} wL = n + 1$,
- $\operatorname{rk} w\mathbb{Q}(t_1)^L = 1$ if $\operatorname{char}(\overline{L}) = p > 0$ and $\operatorname{rk} wL = n + 1$,
- t_{n+1} is a cofinal element in L.

Let $\hat{w}$ be a henselian coarsening of w and $\widehat{\overline{L}}$ the corresponding residue field. In fact we may assume that $\hat{w}L = wL/\Gamma$, where Γ is the largest proper convex subgroup of wL. Note that by the proof of Lemma 3.12 $w\mathbb{Q}(t_1, \dots, t_n)^L \subseteq \Gamma$. So $\operatorname{rk} \hat{w}\mathbb{Q}(t_1, \dots, t_n)^L = 0$ and the residue field $\overline{\mathbb{Q}(t_1, \dots, t_n)^L}$ with respect to the trivial valuation is just $\mathbb{Q}(t_1, \dots, t_n)^L$ itself. Applying Proposition 4.1 with $K = \mathbb{Q}(t_1, \dots, t_n)^L$ we get

$$\operatorname{tr} \deg \widehat{\overline{L}}/\mathbb{Q}(t_1, \dots, t_n)^L + \operatorname{rk} \hat{w}L \le \operatorname{tr} \deg L/\mathbb{Q}(t_1, \dots, t_n)^L = 1.$$

Hence

$$\operatorname{tr} \deg \widehat{\overline{L}}/\mathbb{Q}(t_1, \dots, t_n)^L = 0.$$

That is, $\widehat{\overline{L}}$ is algebraic over $\mathbb{Q}(t_1, \dots, t_n)^L$.

Since $(\mathbb{Q}(t_1, \dots, t_n)^L, w)$ is clearly flat and prohenselian, by the inductive hypothesis the restriction of w to $\mathbb{Q}(t_1, \dots, t_n)^L$ is henselian. Let $(\widehat{\overline{L}}, w')$ be the

valued field induced by the pair $w, \hat{w}$. There is an induced valued-field embedding of $(\mathbb{Q}(t_1, \dots, t_n)^L, w)$ into $(\widehat{\overline{L}}, w')$. But $\widehat{\overline{L}}$ is algebraic over $\mathbb{Q}(t_1, \dots, t_n)^L$ and w is a henselian valuation, clearly w' is also a henselian valuation. Now by Theorem 3.5 we conclude that (L, w) is henselian. $\dashv$

§5. Henselianity and Denef-Pas style languages. Recall that the three component languages of the prototypical Denef-Pas language $\mathcal{L}_{RRP}$ are $\mathcal{L}_R$, $\mathcal{L}_R$, and $\mathcal{L}_{Pr\infty}$. For simplicity, we work with the version of $\mathcal{L}_{Pr}$ that does not contain the inverse function symbol $-$. Throughout this section let $S = \langle K, \overline{K}, \Gamma \cup \{\infty\}, v, \overline{\mathrm{ac}} \rangle$ be a structure of $\mathcal{L}_{RRP}$ such that

1. char $K = 0$,
2. char $\overline{K} = 0$,
3. v and $\overline{\mathrm{ac}}$ are interpreted as a valuation map and an angular component map respectively,
4. the value group Γ is a $\mathbb{Z}$-group,
5. the theory $\mathrm{Th}(S)$ admits QE in the K-sort.

We shall prove:

THEOREM 5.1. *Under these conditions, the valuation v is henselian.*

The proof of this theorem can be adapted for other Denef-Pas style languages as well, provided that the value group satisfies certain mild conditions; see Remark 5.11.

Remark 5.2. The theory of $\mathbb{Z}$-groups with a top element in $\mathcal{L}_{Pr\infty}$ admits QE. This basically follows from Lemma 5.4 and Lemma 5.5, [11].

In this section the following notational conventions are adopted. We use X, Y, etc. for K-sort variables, M, N, etc. for Γ-sort variables, and Ξ, Λ, etc. for $\overline{K}$-sort variables. The lowercase of these letters stands for closed terms or elements in the corresponding sorts. Unless indicated otherwise, all these letters stand for tuples of variables whenever they appear in a formula. We use $\mathrm{lh}\, X$ to denote the length of X. Let $\mathbb{Z}$ and $\overline{\mathbb{Z}}$ be the rings of integers of K and $\overline{K}$, respectively. Let $\mathbb{Z}_\Gamma$ be the smallest convex subgroup of Γ.

Every quantifier-free formula in $\mathcal{L}_{RRP}$ is a disjunction of conjunctions of literals of the following kinds:

- Type A: $F(X) \square 0$, where $\square$ is either $=$ or $\neq$ and $F(X) \in \mathbb{Z}[X]$.
- Type B: $vF_1(X) + M_1 + n_1 \square vF_2(X) + M_2 + n_2$, where $\square$ is one of the symbols $=, \neq, <, >$, and $F_1(X), F_2(X) \in \mathbb{Z}[X]$.
- Type C: $D_h(vF(X)+M+n)$ or $\neg D_h(vF(X)+M+n)$, where $F(X) \in \mathbb{Z}[X]$.
- Type D: $\sum_{i=1}^h G_i(\Lambda) \overline{\mathrm{ac}}\, F_i(X) \square 0$, where $\square$ is either $=$ or $\neq$, $F_i(X) \in \mathbb{Z}[X]$, and $G_i(\Lambda) \in \overline{\mathbb{Z}}[\Lambda]$.

The following lemma is slightly more general than [11, Lemma 5.3]. Recall the definitions concerning Denef-Pas style languages in Section 2.

LEMMA 5.3. *Let φ be a simple formula in $\mathcal{L}_{RRP}$. Then φ is equivalent to a formula of the form*

$$\bigvee_i (\sigma_i \wedge \chi_i \wedge \theta_i)$$

where σ_i is a quantifier-free formula in $\mathcal{L}_K$, χ_i a $\overline{K}$-formula, and θ_i a Γ-formula.

PROOF. We can write φ in its prenex normal form $Q_1 \dots Q_k \; \psi$ where each Q_j is either a Γ-quantifier or a $\overline{K}$-quantifier and ψ is a quantifier-free formula. We proceed by induction on the number k of quantifiers.

If $k = 0$ then φ is quantifier-free. So φ can be written in its disjunctive normal form

$$\bigvee_i (\sigma_i \wedge \chi_i \wedge \theta_i)$$

where σ_i is a conjunction of literals of Type A, χ_i a conjunction of literals of Type D, and θ_i a conjunction of literals of Type B and Type C. This proves the base case.

Suppose now $k = l + 1$. So by the inductive hypothesis φ can be written in the form

$$Q_1 \; \bigvee_i (\sigma'_i \wedge \chi'_i \wedge \theta'_i)$$

where σ'_i is a quantifier-free formula in $\mathcal{L}_K$, χ'_i a $\overline{K}$-formula, and θ'_i a Γ-formula. If Q_1 is $\exists N$ then we can simply push the quantifier in and write φ as

$$\bigvee_i (\sigma'_i \wedge \chi'_i \wedge \exists N \; \theta'_i).$$

If Q_1 is $\forall N$ then we can rewrite $\bigvee_i (\sigma'_i \wedge \chi'_i \wedge \theta'_i)$ in its conjunctive normal form and then push the quantifier in. The other two cases of Q_1 being $\exists \Xi$ or $\forall \Xi$ are treated in the same way. $\dashv$

Simple formulas play an important role in this section. By Remark 5.2 and Lemma 5.3, they can be written as disjunctions of conjunctions of formulas of the following forms:

- Type I: Same as Type A.
- Type II: Same as Type B. Note that, since the conditions $vF(X) = \infty$ and $vF(X) \neq \infty$ are equivalent to the conditions $F(X) = 0$ and $F(X) \neq 0$ respectively and the latter ones can be assimilated into Type I, we may assume that $F(X) \neq 0$ for each $F(X) \in \mathbb{Z}[X]$ that appears in a formula of this type.
- Type III: Same as Type C. As in Type II we may assume that $F(X) \neq 0$ for each $F(X) \in \mathbb{Z}[X]$ that appears in a formula of this type.
- Type IV: $\overline{K}$-formulas; that is, formulas of the form $Q_1 \dots Q_k \; \psi$, where each Q_j is a $\overline{K}$-quantifier and ψ is a disjunction of conjunctions of literals of Type D. Again, since the conditions $\overline{\mathrm{ac}} \, F(X) = 0$ and $\overline{\mathrm{ac}} \, F(X) \neq 0$ are equivalent to the conditions $F(X) = 0$ and $F(X) \neq 0$, we may assume that $F(X) \neq 0$ for each $F(X) \in \mathbb{Z}[X]$ that appears in a formula of this type.

5.1. Step 1: Clopen sets. Since Step 2 and Step 3 do not involve formulas that contain free $\overline{K}$-variables or free Γ-variables, we may limit our attention to such formulas of Type I, II, III, and IV. We shall show that such formulas, except the equalities in the K-sort, define open sets in the corresponding product of the valuation topology. This takes care of Step 1 in Section 2.

Since quotients of polynomials are continuous maps with respect to the valuation topology, that formulas of Type II define clopen sets follows from the basic fact that, for $m \in \Gamma$, sets of the forms $\{x : v(x) = m\}$, $\{x : v(x) > m\}$, etc. are all clopen in the valuation topology. See [6, Remark 2.3.3].

LEMMA 5.4. *Let $\varphi(X)$ be a formula of Type III. Then φ defines a clopen set.*

PROOF. First let $\varphi(X)$ be of the form $D_h(vF(X) + n)$. Let $B \subseteq \Gamma$ be the set of all solutions of the formula; that is, $m \in B$ if and only if $S \models D_h(m + n)$. For each $m \in \Gamma$ let

$$A_m = \left\{ x \in (K^\times)^e : vF(x) = m \right\},$$

where $e = \mathrm{lh}\, X$. Since polynomial maps are continuous, each A_m is clopen in the valuation topology. So

$$\varphi((K^\times)^e) = \bigcup_{m \in B} A_m = (K^\times)^e \setminus \bigcup_{m \notin B} A_m$$

is clopen. The other case follows immediately from this. ⊣

Let $\mathcal{O}, \mathcal{M}$ be the valuation ring and its maximal ideal that correspond to v. The following lemma establishes a crucial relation between the valuation and the angular component map.

LEMMA 5.5. *For nonzero $x, y \in K$ with $v(x) = v(y) = m \in \Gamma$, $\overline{ac}\, x = \overline{ac}\, y$ if and only if $v(x - y) > m$.*

PROOF. If $x = y$ then the lemma is trivial. So we assume further that $x \neq y$.
For the "only if" direction, suppose for contradiction that $\overline{ac}\, x = \overline{ac}\, y$ but $v(x - y) = m$. So $(x - y)/x$ is a unit. So

$$\overline{ac}\, \frac{x - y}{x} = 1 - \frac{y}{x} + \mathcal{M}$$
$$= 1 + \mathcal{M} - \left(\frac{y}{x} + \mathcal{M} \right)$$
$$= 1 + \mathcal{M} - \overline{ac}\, \frac{y}{x}$$
$$= 1 + \mathcal{M} - \frac{\overline{ac}\, y}{\overline{ac}\, x}$$
$$= 0.$$

So $(x - y)/x = 0$, so $x = y$, contradiction.
For the "if" direction, suppose for contradiction that $v(x - y) > m$ but $\overline{ac}\, x \neq \overline{ac}\, y$. If $m = 0$, that is, x and y are units in the valuation ring, then

$$x + \mathcal{M} = \overline{ac}\, x \neq \overline{ac}\, y = y + \mathcal{M}.$$

So $x - y$ is a unit in the valuation ring, that is, $v(x - y) = 0$, contradiction. In general we may consider $1 - y/x$: since $v(1 - y/x) > 0$ and y/x is a unit, we get $\overline{ac}\, 1 = \overline{ac}(y/x)$ by the previous two sentences, so $\overline{ac}\, x = \overline{ac}\, y$. ⊣

LEMMA 5.6. *Let $\zeta \in \overline{K}^\times$ and $F(X) \in \mathbb{Z}[X]$. The set*

$$A_\zeta = \left\{ x \in (K^\times)^e : \overline{ac}\, F(x) = \zeta \right\}$$

is clopen, where $e = \mathrm{lh}\, X$.

PROOF. Let $X = \langle X_1, \dots, X_e \rangle$. Write $F(X)$ as $\sum_i f_i G_i(X)$, where $f_i \in \mathbb{Z}$ and each $G_i(X)$ is a unique monomial in the summation. Let c be a natural number that is larger than all the exponents of the variables that appear in $F(X)$. Let $x = \langle x_1, \dots, x_e \rangle \in (K^\times)^e$. For each $n \in \Gamma$ let $|n| = n$ if $n \geq 0$, otherwise $|n| = -n$. For each x_j with $1 \leq j \leq e$ let

$$U_j = \{x_j + y : y \in K \text{ and } v(y) > vF(x) + c\,|v(x_1)| + \dots + c\,|v(x_e)|\}.$$

Note that $x_j \in U_j$ and $0 \notin U_j$. Clearly each U_j is clopen in the valuation topology. Let

$$U_x = U_1 \times \dots \times U_e.$$

Now each $G_i(X)$ is of the form

$$X_1^{c_1} \cdots X_e^{c_e}.$$

For any $\langle x_1 + y_1, \dots, x_e + y_e \rangle \in U_x$ we have

$$f_i(x_1 + y_1)^{c_1} \cdots (x_e + y_e)^{c_e} = f_i x_1^{c_1} \cdots x_e^{c_e} + H(x, y),$$

where $y = \langle y_1, \dots, y_e \rangle$, $H(X, Y) \in \mathbb{Z}[X, Y]$, and, by the choice of U_x,

$$vH(x, y) > vF(x).$$

So

$$vF(x_1 + y_1, \dots, x_e + y_e) = vF(x)$$

and

$$v(F(x_1 + y_1, \dots, x_e + y_e) - F(x)) > vF(x).$$

So by Lemma 5.5 we get

$$\overline{\mathrm{ac}}\, F(x_1 + y_1, \dots, x_e + y_e) = \overline{\mathrm{ac}}\, F(x).$$

So

$$A_\zeta = \bigcup_{x \in A_\zeta} U_x = (K^\times)^e \setminus \bigcup_{x \notin A_\zeta} U_x$$

is clopen.   ⊣

Remark 5.7. It may seem that we can use the continuity of polynomial maps, much as in the proof of Lemma 5.4, to prove the above lemma. But this does not work because for $\zeta \in \overline{K}^\times$ the set

$$A_\zeta = \{x \in K : \overline{\mathrm{ac}}\, x = \zeta\},$$

although clopen in the valuation topology on $K^\times$, is not closed in the valuation topology on K as there is no open neighborhood of 0 that does not intersect with A_ζ. This is the reason why we have chosen to work with the valuation topology on $K^\times$ instead of K.

LEMMA 5.8. *Let $\varphi(X)$ be a formula of Type IV. Then φ defines a clopen set.*

PROOF. Let $\varphi(X)$ be of the form $Q_1 \ldots Q_k \ \psi(X)$ where $\psi(X)$ is a disjunction of conjunctions of formulas of the form $\sum_{i=1}^{h} \lambda_i \overline{\mathrm{ac}} \, F_i(X) + \xi \ \square \ 0$ with $\lambda_i, \xi \in \overline{\mathbb{Z}}$. Let $B \subseteq (\overline{K}^{\times})^h$ be the set of all solutions of the formula; that is, $\langle \zeta_i \rangle \in B$ if and only if

$$S \models Q_1 \ldots Q_k \ \psi^*(\langle \zeta_i \rangle),$$

where the formula $\psi^*(\langle \zeta_i \rangle)$ is obtained by replacing each $F_i(X)$ in $\psi(X)$ with ζ_i. For each $\langle \zeta_i \rangle \in (\overline{K}^{\times})^h$ let

$$A_{\langle \zeta_i \rangle} = \bigcup \bigcap \left\{ x \in (K^{\times})^e : \overline{\mathrm{ac}} \, F_i(x) = \zeta_i \right\}$$

be the boolean combination of sets that corresponds to $\psi^*(\langle \zeta_i \rangle)$, where $e = \mathrm{lh} \, X$. By Lemma 5.6 each $A_{\langle \zeta_i \rangle}$ is clopen. So

$$\varphi((K^{\times})^e) = \bigcup_{\langle \zeta_i \rangle \in B} A_{\langle \zeta_i \rangle} = (K^{\times})^e \setminus \bigcup_{\langle \zeta_i \rangle \notin B} A_{\langle \zeta_i \rangle}$$

is clopen. $\dashv$

5.2. Step 3: Omitting a type. For the rest of this section let X, Y be two single variables. To carry out Step 3 in Section 2 we will again omit the 2-type (3.1) to show:

THEOREM 5.9. *There is a structure $S_1 = \langle K_1, \overline{K}_1, \Gamma_1 \cup \{\infty\}, v_1, \overline{\mathrm{ac}}_1 \rangle$ of $\mathcal{L}_{RRP}$ such that $S_1 \equiv S$ and v_1 is of rank 1.*

LEMMA 5.10. *Let $\varphi(X, Y)$ be a conjunction of formulas of Type II and III, where X, Y are the only free variables. Let $x, y \in \mathcal{M}$ be nonzero such that $x \ll y$ and $S \models \varphi(x, y)$. Then for every natural number k there is an $m \in \Gamma$ with $v(x^k) < m < v(x^l)$ for some $l > k$ such that for every $t \in \mathcal{M}$ with $v(t) = m$ we have*

$$S \models \varphi(x, t).$$

PROOF. Let $F_i(X, Y) \in \mathbb{Z}[X, Y]$ run through all the distinct polynomials that appear in $\varphi(X, Y)$. We may assume that each $F_i(X, Y)$ is written in the form (3.3) and (3.4). It is not hard to see that if we choose a $k_0 > 0$ that is larger than the sum of all the exponents of X that appear in all the $F_i(X, Y)$'s, then, for each nonzero $t \in \mathcal{M}$, if $v(t) > v(x^{k_0})$ then

(5.1) $$v(F_i(x, t)) = v(x^{e_i} t^{d_i})$$

for some integers $e_i, d_i \geq 0$ with $e_i < k_0$. Clearly in this situation e_i, d_i are independent of the choice of t. Substituting two free variables N_1, N_2 for $v(x), v(t)$ respectively we may rewrite $\varphi(x, t)$ as a formula $\varphi^*(N_1, N_2)$ in $\mathcal{L}_{Pr\infty}$. So we have

$$\Gamma \cup \{\infty\} \models \varphi^*(v(x), v(y)).$$

Now let $v(x) = n$. Let $\Gamma(n)$ be the smallest $\mathbb{Z}$-group generated by n in Γ. It is easy to see that the set $\{kn : k \in \mathbb{N}\}$ is cofinal in $\Gamma(n)$. Clearly $\Gamma(n) \cup \{\infty\}$ is an elementary substructure of $\Gamma \cup \{\infty\}$. So for every natural number $k \geq k_0$ we have

$$\Gamma(n) \cup \{\infty\} \models \exists N \ (kn < N < \infty \wedge \varphi^*(n, N)).$$

So for some $m \in \Gamma(n)$ and some $l > k$ we have

$$\Gamma(n) \cup \{\infty\} \models kn < m < ln \wedge \varphi^*(n, m).$$

So for every $t \in \mathcal{M}$ with $v(t) = m$ we have

$$\Gamma \cup \{\infty\} \models \varphi^*(n, v(t)).$$

By the choice of k_0 this clearly implies that

$$S \models \varphi(x, t),$$

as desired. $\dashv$

Remark 5.11. A close examination of the proof of Lemma 5.4 shows that, much as Lemma 5.8, regardless of what language the group Γ uses and what additional structure it has, Γ-formulas without free Γ-variables always define clopen sets. Therefore Lemma 5.10 is actually the only place where we need to use some special properties that hold in $\mathbb{Z}$-groups, namely

1. for any element n in the Γ-sort the set $\{kn : k \in \mathbb{N}\}$ is cofinal in the submodel generated by n;
2. the theory of the Γ-sort in $\mathcal{L}_{\Gamma\infty}$ is model-complete.

So our converse QE result holds for any group Γ and any language $\mathcal{L}_\Gamma$ such that these two properties are satisfied.

LEMMA 5.12. *Let $\varphi(X, Y)$ be a formula of Type IV, where X, Y are the only free variables. Let $x, y \in \mathcal{M}$ be nonzero such that $x \ll y$ and $S \models \varphi(x, y)$. For every sufficiently large natural number k, if $t \in \mathcal{M}$ is such that $v(t) \geq v(x^k)$ and $\overline{ac}\, t = \overline{ac}\, y$ then*

$$S \models \varphi(x, t).$$

PROOF. Let $F_i(X, Y) \in \mathbb{Z}[X, Y]$ run through all the distinct polynomials that appear in $\varphi(X, Y)$. As in the previous lemma we may choose a $k > 0$ that is larger than the sum of all the exponents of X that appear in all the $F_i(X, Y)$'s so that for each nonzero $t \in \mathcal{M}$, if $v(t) > v(x^k)$ then the condition (5.1) holds for each $F_i(X, Y)$. For such a $t \in \mathcal{M}$, if $F_i(X, Y)$ is written in the form (3.3) and (3.4), then we have

$$v(F_b(x)t^b + \ldots + F_0(x)) = vF_0(x)$$

and

$$v(F_b(x)t^b + \ldots + F_1(x)t) > vF_0(x)$$

if $b > 0$, where $F_0(X)$ is written as

$$X^f(s_a X^a + \ldots + s_0),$$

with $s_0, \ldots, s_a \in \mathbb{Z}$ and s_0 nonzero. So by Lemma 5.5 we have

$$\overline{ac}\, F_i(x, t) = \overline{ac}(t^d(F_b(x)t^b + \ldots + F_0(x))) = (\overline{ac}\, t)^d \cdot \overline{ac}\, F_0(x)$$

and

$$\overline{ac}\, F_0(x) = (\overline{ac}\, x)^f \cdot \overline{ac}\, s_0.$$

In particular, since $x \ll y$, we have

$$\overline{ac}\, F_i(x, y) = (\overline{ac}\, x)^f \cdot (\overline{ac}\, y)^d \cdot \overline{ac}\, s_0.$$

Now if $\overline{\mathrm{ac}}\, t = \overline{\mathrm{ac}}\, y$ then we have

$$\overline{\mathrm{ac}}\, F_i(x,t) = (\overline{\mathrm{ac}}\, x)^f \cdot (\overline{\mathrm{ac}}\, t)^d \cdot \overline{\mathrm{ac}}\, s_0 = (\overline{\mathrm{ac}}\, x)^f \cdot (\overline{\mathrm{ac}}\, y)^d \cdot \overline{\mathrm{ac}}\, s_0 = \overline{\mathrm{ac}}\, F_i(x,y).$$

So clearly

$$S \models \varphi(x,t),$$

as desired. $\dashv$

LEMMA 5.13. *The 2-type* $\Phi(X,Y)$ *is not isolated modulo* $\mathrm{Th}(S)$.

PROOF. Suppose for contradiction that there is a formula $\pi(X,Y)$ such that

- $\exists X, Y\ \pi(X,Y) \in \mathrm{Th}(S)$, and
- $\pi(X,Y) \vdash \Phi(X,Y)$ modulo $\mathrm{Th}(S)$.

Since $\mathrm{Th}(S)$ admits QE in the K-sort, by Lemma 5.3, $\pi(X,Y)$ is equivalent to a disjunction of conjunctions of formulas of Type I, II, III, and IV. Without loss of generality we may assume that $\pi(X,Y)$ is just a conjunction of formulas of those four types. Let $x \ll y$ be such that $S \models \pi(x,y)$. We shall show that there is a $t \in \mathcal{M}$ with $x \asymp t$ such that

$$S \models \pi(x,t).$$

This yields a contradiction.

By Lemma 3.12 $\pi(X,Y)$ cannot contain equalities in the K-sort. Clearly, for sufficiently large k, if $t \in \mathcal{M}$ is nonzero and $v(t) \geq v(x^k)$ then the pair (x,t) satisfies the inequality in the K-sort that appear in $\pi(X,Y)$. Finally, by Lemma 5.10 and 5.12 we can choose a sufficiently large k and a $t \in \mathcal{M}$ with $v(x^k) < v(t) < v(x^l)$ for some $l > k$ and $\overline{\mathrm{ac}}\, t = \overline{\mathrm{ac}}\, y$ such that $S \models \pi(x,t)$, as desired. $\dashv$

Now Theorem 5.9 follows immediately from this lemma and the Omitting Types Theorem.

Remark 5.14. It is not hard to see that, by considering the formula $\exists X\ \pi(X,Y)$ as in Lemma 3.18 and Theorem 3.19, the proofs in this section can be modified to cover the case that $\mathrm{char}(\overline{K}) = p > 0$ and (K,v) is tight. We no longer need the condition that there is a conservative subgroup since now the theory of the Γ-sort is already model-complete.

§6. **Naturality of language.** In this section we describe a general perspective on QE and converse QE results. This concerns the usually vague notion that a language is "natural" for a mathematical structure. Here we propose a precise criterion of naturality by which a language $\mathcal{L}$ can be judged with respect to a chosen property P:

CRITERION 6.1. Modulo some basic properties (to be specified in context), $\mathcal{L}$ is natural with respect to P if and only if any structure of $\mathcal{L}$ that has P admits QE in $\mathcal{L}$ and any structure of $\mathcal{L}$ that admits QE in $\mathcal{L}$ has P.

In other words, $\mathcal{L}$ is natural with respect to P if and only if QE in $\mathcal{L}$ characterizes P. Hence in order to show that $\mathcal{L}$ is natural with respect to P one has to show QE and converse QE.

By Tarski's Theorem and Theorem 1.3, modulo the defining properties of ordered fields, the language $\mathcal{L}_{OR}$ is natural with respect to real-closedness. Similarly by Pas's QE result in [11] and Theorem 5.1, modulo the other properties presented at the beginning of Section 5, the language $\mathcal{L}_{RRP}$ is natural with respect to henselianity. However, we need the extra conditions described in Section 3 to establish the naturality of $\mathcal{L}_{Mac}$ or $\mathcal{L}_{Mac,D}$.

Let us examine a simpler case: the $\mathbb{Z}$-groups. A $\mathbb{Z}$-group is a group that is elementarily equivalent to the group $\mathbb{Z}$ of the integers in the Presburger language $\mathcal{L}_{Pr}$. By Presburger's Theorem the theory of $\mathbb{Z}$-groups admits QE in $\mathcal{L}_{Pr}$. The proof uses the condition that 1 is the least positive element. However, by Criterion 6.1, modulo everything else in the theory, $\mathcal{L}_{Pr}$ is not natural with respect to this condition. This is a consequence of [14, Corollary 2.11] which implies that the structure

$$\langle \mathbb{Z} \times \mathbb{Q}, +, -, <, 0, 1, D_n \rangle_{n>1}$$

admits QE, where everything is interpreted in the standard way except that $<$ is the lexicographic ordering and the constant 1 designates the element $(1, 0)$.

What about other properties of $\mathbb{Z}$-groups? In every $\mathbb{Z}$-group, for each divisibility predicate D_n, the following holds:

$$(6.1) \qquad \forall x \, (D_n(x) \vee D_n(x+1) \vee \ldots \vee D_n(x+n-1)).$$

This property is also needed for Presburger's Theorem. Now if we weaken Criterion 6.1 by requiring P to be equivalent to model-completeness (that is, every formula is equivalent to an existential formula), then $\mathcal{L}_{Pr}$ is again not natural with respect to the property (6.1). This is again a consequence of [14, Corollary 2.11] which also implies that the structure

$$\langle \mathbb{Z} \times \mathbb{Z}, +, -, <, 0, 1, 1', D_n \rangle_{n>1}$$

admits QE, where $<$ is again the lexicographic ordering, the constant 1 designates the least positive element $(0, 1)$, and the constant $1'$ designates the element $(1, 0)$. Note that $(1, 0)$ is not definable with the rest of the structure. But, instead of $(1, 0)$, any element that satisfies the formula

$$\forall x \bigvee_{i,j<n} D_n(x + i \cdot y + j \cdot 1)$$

for each $n > 1$ can be used in the QE procedure. So every formula is equivalent to an existential formula in the reduct of the structure to $\mathcal{L}_{Pr}$.

Question 6.2. Is the language $\mathcal{L}_{Pr}$ natural with respect to the property (6.1) by Criterion 6.1? That is, is there a commutative group with discrete total ordering that admits QE in $\mathcal{L}_{Pr}$ but does not satisfy the property (6.1)?

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